Study materials of Mathematics for class D-III (H), Paper-VII on the topic " D'Alembert Principle" of dynamics, Composed by Dr. Md. Shamim Ahmad, Associate Professor, U.R. College Rosera.

D'Alembert's Principle:-

Let x, y, z be the Co-ordinates at time t of a particle of mass m in the body.

Then $m\ddot{x}, m\ddot{y}, m\ddot{z}$ are called the Components of effective force (or Kinetic reaction) acting on the particle parallel to the axes of x, y, z respectively. By Newton's Third Law of motion

$$m\ddot{x} = F_x + I_x$$
$$m\ddot{y} = F_y + I_y$$
$$m\ddot{z} = F_z + I_z$$

Where F_x, F_y, F_z are the Components of external forces and I_x, I_y, I_z the Components of internal forces acting on the particle.

Thus

$F_x - m\ddot{x} + I_x$, $F_y - m\ddot{y} + I_y$, $F_z - m\ddot{z} + I_z$ are in equilibrium taking over all particles composing the body. But $\Sigma I_x, \Sigma I_y, \Sigma I_z$ are together in equilibrium themselves by Newton's Third Law every action is equal and opposite reaction. Hence

$F_x - m\ddot{x}$, $F_y - m\ddot{y}$, $F_z - m\ddot{z}$

taken over all the particles of the body form a system in equilibrium.

This result is known as D'Alembert's Principle. It states that "$\boxed{\text{the reversed effective forces and the external forces acting on all particles of the body form a system in equilibrium}}$".

Equation of motion :-

By the conditions of equilibrium from statics we have the following equations:

$$\sum (F_x - m\ddot{x}) = 0, \quad \sum (F_y - m\ddot{y}) = 0, \quad \sum (F_z - m\ddot{z}) = 0$$

and
$$\sum \left[y(F_z - m\ddot{z}) - z(F_y - m\ddot{y}) \right] = 0$$

$$\sum \left[z(F_x - m\ddot{x}) - x(F_z - m\ddot{z}) \right] = 0$$

$$\sum \left[x(F_y - m\ddot{y}) - y(F_x - m\ddot{x}) \right] = 0$$

Thus we get

$$\left.\begin{aligned} \sum m\ddot{x} &= \sum F_x \\ \sum m\ddot{y} &= \sum F_y \\ \sum m\ddot{z} &= \sum F_z \end{aligned}\right] \text{—— (1.)}$$

and
$$\left.\begin{aligned} \sum m(y\ddot{z} - z\ddot{y}) &= \sum (yF_z - zF_y) \\ \sum m(z\ddot{x} - x\ddot{z}) &= \sum (zF_x - xF_z) \\ \sum m(x\ddot{y} - y\ddot{x}) &= \sum (xF_y - yF_x) \end{aligned}\right] \text{—— (2)}$$

The equations (1) and (2) Constitute "The Equation of Motion" of a rigid body

Also, The equations (1) state that "the sum of components parallel to the Co-ordinate axes of the effective forces are respectively equal to the sums of the components of the external forces parallel to the same axes" — 1 (A)

Also, the equations (2) state that " sums of moments about the Co-ordinate axes of the effective forces are respectively equal to the sums of moments of the external forces about the same axes."

— 1 (B)

Again, since $\sum m\ddot{x}$, $\sum m\ddot{y}$, $\sum m\ddot{z}$ are the momentum of the system parallel to x, y, z-axes respectively and $\sum m\ddot{x} = \frac{d}{dt}(\sum m\dot{x})$ etc. The notes 1(A) and 1(B) Contain the following ~~facts~~ fundamental facts:

(A′) The rate of change of momentum of a body in any given direction is equal to the resolved part of the external forces in the same direction.

(B′) The moment of the rate of change of momentum about any fixed axis is equal to the sum of the moments of the external forces about the same axis.